Equilibrium statistical mechanics of one-dimensional Hamiltonian systems with long-range force

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The system of N identical classical particles on the circle of length L interacting via a pair potential is investigated in the mean field limit ($N\to\infty$, L fixed). Its physical properties are determined by the Fourier components of the interaction (mean) field. The partition function, the joint distribution of the interaction fields, the local field, and the correlation functions are computed. If the interaction is semidefinite non-negative, field components become independent for $N\to\infty$ and satisfy central limit theorems. If the interaction has negative Fourier components, a phase transition occurs. [S1063-651X(97)15705-1]

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Systems of many particles interacting via two-body long-range forces have peculiar equilibrium and nonequilibrium statistical mechanics [1,2]. The Coulomb system, which has been thoroughly investigated, combines the difficulties of long-range interaction with a short-range divergence—or, in one dimension, with a discontinuity in the electric field. Recent numerical simulations [3–6] and theoretical arguments [7–9] indicate that the characteristics of Coulombian plasma turbulence are shared by a family of systems in which only the long-range part of the Coulomb potential is kept; this opens a way to faster molecular-dynamics simulations of these systems. Similarly, the gravitational system, which is equally fundamental, exhibits specific statistical behaviors [10–12]. A truncation of the gravitational interaction to long-range components was also seen to preserve some of its dynamical relaxation characteristics [13,14].

A proper understanding of the analogy and differences between systems with long-range pair interactions requires a discussion of their equilibrium statistical mechanics. The aim of the present paper is to show which systems with long-range interactions behave similarly to the Coulomb one and which systems do not—in particular by undergoing a phase transition. For simplicity we restrict our discussion to one-dimensional models [15].

In Sec. I we introduce the model family and its relevant variables; we discuss the connection between the mean-field limit ($N\to\infty$, L fixed) and the thermodynamic limit ($L\to\infty$, N/L fixed). In Sec. II we reduce the computation of canonical averages to the estimation of Bessel-like integrals. Their evaluation is straightforward for the plasma family as shown in Sec. III. In Sec. IV we comment on the phase transition induced by attractive components in the interaction potential, which was also discussed in a single-component case by Inagaki and co-workers [13,14] and by Ruffo and co-workers [3,16].

I. HAMILTONIAN ROTATOR MODEL

In the present paper, we discuss the equilibrium properties of a system of N identical classical particles described by the Hamiltonian

$$H=\sum_{j=1}^{N}\frac{p_j^2}{2m_j}+V_{\rm tot}, \tag{1}$$

with

$$V_{\rm tot}=\frac{1}{2N}\sum_{j,r=1}^{N} q_r q_j V(x_r-x_j), \tag{2}$$

$$V(y)=\sum_{n=1}^{s} V_n \cos k_n y, \tag{3}$$

where x_j is the position of particle j (with mass m_j and charge q_j) on the interval of length L with periodic boundary conditions (viz. the circle $S_L=\mathbb{R}/L$), $p_j\in\mathbb{R}$ is its conjugate momentum, and $k_n=2\pi n/L$. So far, interest in the equilibrium statistical mechanics of such one-dimensional gas models was motivated principally by (i) the rigorous understanding of phase transitions and of equations of state of van der Waals type [17–20], (ii) the search for exactly solvable models, and (iii) the specific case of Coulomb and gravitational interactions. Well-known models include the following.

(1) The ideal gas of noninteracting particles: $V_n=0$ for all n.

(2) The mean-field Hamiltonian antiferromagnetic XY model [3]: $V_1>0$ and all $V_n=0$ for $n>1$.

(3) The Coulomb plasma on the circle without neutralizing background of Lenard and Prager [4,6–9,21,22]: $V_n=n^{-2}K'$ for n running over odd positive integers, with $s\to\infty$ and $K'=L/(\pi^2\epsilon_0)>0$, so that $V(x)=(\frac{1}{4}-|\xi|)L/(2\epsilon_0)$, $\xi=x/L(\mathrm{mod}1)\in[-1/2,1/2]$.

(4) The jellium (or one-component Coulomb plasma with uniform neutralizing background) on the circle [23,24]: $V_n=n^{-2}K'$ for n running over all positive integers, with $s\to\infty$ and $K'=L/(2\pi^2\epsilon_0)>0$, so that $V(x)=(\frac{1}{6}-|\xi|+\xi^2)L/(2\epsilon_0)$.

(5) The single-species ($q_r=1\ \forall r$) exponentially repulsive points model [25]: $V_n=K'\gamma L/(n^2+\gamma^2L^2)$, with $K'>0$ and $\gamma>0$, so that $V(x)=K'\sum_{r=-\infty}^{\infty}\exp(-\gamma|x-rL|)$.

(6) The hard rod model [26] as a limit of the previous model with $\gamma\to\infty$ and $K'e^{\gamma b}$ fixed (with $b>0$ the rod length).

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(7) Charged hard rods [27].

(8) The single-species mean-field Hamiltonian ferromagnetic XY model [3,13,14,16]: $V_1<0$ and all $V_n=0$ for $n>1$.

(9) The single-species self-gravitating system on the circle: $V_n=-n^{-2}\pi^{-2}LG$ for n running over odd positive integers, with $s\to\infty$ and gravitation constant $G>0$.

The obvious constants of the motion are total momentum

$$P=\sum_{j=1}^{N} p_j \tag{4}$$

and the energy $E=H$. The Hamiltonian (1) generates the equations of motion

$$\dot{x}_j=p_j/m_j, \tag{5}$$

$$\dot{p}_j=\frac{q_j}{N}\sum_{n=1}^{s}\sum_{r=1}^{N} q_r k_n V_n \sin k_n(x_j-x_r), \tag{6}$$

which are more conveniently expressed using the mean-field variables $\mathbf{S}_n$, defined as

$$\mathbf{S}_n=\left(\frac{1}{N}\sum_{j=1}^{N} q_j\cos k_n x_j, \frac{1}{N}\sum_{j=1}^{N} q_j \sin k_n x_j\right), \tag{7}$$

with amplitude $S_n=|\mathbf{S}_n|$ and phase $\phi_n=\mathrm{Arg}(\mathbf{S}_n)$, with $-\pi<\phi_n\leqslant\pi$. The two components of $\mathbf{S}_n$ are the nth Fourier coefficients of the spatial distribution, and correspond to spatial scale L/n. When the distribution of the particles is random uniform on the circle, the central limit theorem implies that

$$S_n=O(N^{-1/2}). \tag{8}$$

Conversely, one expects $S_n=O(1)$ in an ordered (i.e., *clustered*) phase.

Using these fields, the equation of motion (6) reads

$$\dot{p}_j=q_j A(x_j), \tag{9}$$

with the local field

$$A(x)=\sum_{n=1}^{s} k_n V_n S_n \sin(k_n x-\phi_n), \tag{10}$$

where both S_n and ϕ_n depend on time [cf. Eqs. (7)]. The N-body motion thus reduces to a single-particle problem in the self-consistent fields $\mathbf{S}_n$ with $n=1,\ldots,s$. The time evolution of these fields, which follows from Eqs. (5) and (7), is not autonomous, and results from the motions of all particles (but an approximate dynamics involving only resonant particles can be derived in some cases [7–9]).

Our aim in this paper is to investigate the Gibbs canonical distribution of the fields $\mathbf{S}_n$ along with the thermodynamic potentials of our system. This is natural as the potential energy (3) can be written in terms of the mean fields only,

$$V_{\mathrm{tot}}=\frac{N}{2}\sum_{n=1}^{s} V_n S_n^2. \tag{11}$$

Note that this expression is bounded below by $V_{\mathrm{low}}=(N/2)\Sigma_{(m:V_m<0)}V_m Q^2$ with the average quadratic charge

$$Q^2=N^{-1}\sum_j q_j^2. \tag{12}$$

The lower bound V_{low} may be the ground-state energy as in the antiferromagnetic XY model. However, frustration may prevent V_{tot} from reaching V_{low}: consider, e.g., $V_3<0$, $V_5<0$ with $V_1>0$.

It is also important to remark that we keep L fixed in this paper. The scaling by N of the interparticle coupling emphasizes the mean-field limit $N\to\infty$ with all V_n fixed. This is generally not the usual thermodynamic limit, where both the pair interaction potential $V(x)$ and density N/L are fixed. Indeed, in the usual thermodynamic limit, the Fourier representation of the potential $V(y)$ involves L-dependent wave numbers $k_n=2\pi nL^{-1}$. However, mean-field limit and thermodynamic limit are equivalent for the Coulomb-jellium-gravitation family, for which $V_n\propto Nn^{-2}$; in this case, the thermodynamic limit coincides with the Kac or van der Waals limit [19].

II. PARTITION FUNCTION AND DISTRIBUTION OF THE MEAN FIELDS

In the canonical ensemble with inverse temperature β, the partition function is

$$Z(N,\beta)=Z_V Z_K, \tag{13}$$

where one readily finds the kinetic contribution

$$Z_K=\int_{\mathbb{R}^N}\exp\left(-\beta\sum_{j=1}^{N}\frac{p_j^2}{2m_j}\right)d^N p=\prod_{j=1}^{N}\left(\frac{2\pi m_j}{\beta}\right)^{1/2}$$
$$=\left(\frac{2\pi M}{\beta}\right)^{N/2}, \tag{14}$$

with the average mass $M=(\Pi_{j=1}^N m_j)^{1/N}$.

The potential contribution

$$Z_V=\int_{S_L^N}\exp\left(-\beta\frac{N}{2}\sum_{n=1}^{s} V_n S_n^2\right)d^N x \tag{15}$$

is evaluated using the integral representation of Gaussian functions for $\mathbf{g}\in\mathbb{R}^2$ and $c\in\mathbb{R}$:

$$\exp(-cg^2)=\frac{1}{\pi}\int_{\mathbb{R}^2}\exp(-u^2+2i\sqrt{c}\,\mathbf{u}\cdot\mathbf{g})d^2\mathbf{u}, \tag{16}$$

where $g=|\mathbf{g}|$. Substituting definition (7) into Eq. (15), defining

$$c_n=\beta V_n/2, \tag{17}$$

and applying Eq. (16), yields

$$Z_V=\int_{S_L^N}\int_{\mathbb{R}^{2s}}\pi^{-s}\prod_{n=1}^{s}\exp(-u_n^2+2i\sqrt{Nc_n}\mathbf{u}_n\cdot\mathbf{S}_n)d^{2s}\mathbf{u}d^Nx \tag{18}$$

$$=\pi^{-s}L^N\int_{\mathbb{R}^{2s}}\left(\prod_{n=1}^{s}e^{-u_n^2}\right)\prod_{j=1}^{N}(\mathcal{J}(2q_j\sqrt{c/N}\mathbf{u}))d^{2s}\mathbf{u}, \tag{19}$$

where $\sqrt{c/N}\mathbf{u}=(\sqrt{c_n/N}\mathbf{u}_n)$. We define the function of $\mathbf{v}\in\mathbb{C}^{2s}$:

$$\mathcal{J}(\mathbf{v})=\frac{1}{2\pi}\int_{S_{2\pi}}\prod_{n=1}^{s}e^{iv_n\cos(nx'-\theta_n)}dx'=\mathcal{J}(-\mathbf{v})^*, \tag{20}$$

where $\mathbf{v}_n=(v_n\cos\theta_n,v_n\sin\theta_n)$, and the star denotes complex conjugation. Estimating the partition function now reduces to the study of the function $\mathcal{J}$, for which elementary calculus yields a series representation in products of Bessel functions. Thermodynamic potentials then follow directly,

$$F=-\lim_{N\to\infty}\frac{1}{N\beta}\ln Z,\quad U=\frac{\partial(\beta F)}{\partial\beta}. \tag{21}$$

To describe the distribution of mean fields, we compute their joint characteristic function, i.e., the Fourier transform of the joint probability density for the s-component vector $\mathbf{S}$. Since this vector may include components for which $V_n=0$, we actually consider all mean fields $\mathbf{S}_n$ for any finite set of values of n. Thus we compute, for $\boldsymbol{\sigma}\in\mathbb{R}^{2s}$,

$$\Psi(\boldsymbol{\sigma})=\left\langle\prod_{n=1}^{s}e^{i\boldsymbol{\sigma}_n\cdot\mathbf{S}_n}\right\rangle=Z_V^{-1}\pi^{-s}L^N\int_{\mathbb{R}^{2s}}\left(\prod_{n=1}^{s}e^{-u_n^2}\right)\times\prod_{j=1}^{N}\left(\mathcal{J}\left(q_j\frac{\boldsymbol{\sigma}}{N}+2q_j\sqrt{c/N}\mathbf{u}\right)\right)d^{2s}\mathbf{u}. \tag{22}$$

The moments of $\mathbf{S}$ are the coefficients of the Taylor series expansion of Ψ.

For a single species system, the correlation function of the particles is the density at distance y from a test particle; it is given by

$$C(y)=\langle\delta(x_1-x_2-y)\rangle=\frac{1}{L}+\frac{c(y)}{N-1}=\frac{1}{L}+\frac{2}{L(N-1)}\sum_{n=1}^{\infty}C_n\cos k_ny, \tag{23}$$

where the sum extends over all positive integers, with $C_n=Nq^{-2}\langle|\mathbf{S}_n|^2\rangle-1$. Given a particle at $x_1=0$, its contribution to the mean fields (with the bare value q/N) is shielded or enhanced by the other particles, so that the expectations of effective fields have amplitude

$$S_n^{\text{eff}}=\frac{q}{N}(1+LC_n) \tag{24}$$

and phase $\phi_n^{\text{eff}}=0$.

III. PLASMA (OR ANTIFERROMAGNETIC) MODELS

The case where V is *non-negative-definite* [19], i.e., no (*Fourier*) coefficient V_n is (strictly) negative, is easily treated. Indeed, for real vectors $\mathbf{v}\in\mathbb{R}^{2s}$, $|\mathcal{J}(\mathbf{v})|\leq 1$, with the bound reached only at the origin. Therefore, in the limit $N\to\infty$, one expands $\mathcal{J}(\mathbf{v})$ around this maximum to evaluate Eq. (19):

$$Z_V\approx\pi^{-s}L^N\int_{\mathbb{R}^{2s}}\left(\prod_{n=1}^{s}e^{-u_n^2}\right)\left(\prod_{n=1}^{s}e^{-\beta Q^2V_nu_n^2/2}\right)d^{2s}\mathbf{u} \tag{25}$$

$$=L^N\prod_{n=1}^{s}(1+\beta Q^2V_n/2)^{-1}. \tag{26}$$

Similarly the joint characteristic function of mean-field variables is

$$\Psi(\boldsymbol{\sigma})\approx\prod_{n=1}^{s}e^{-Q^2\sigma_n^2/(4+2\beta Q^2V_n)N}. \tag{27}$$

The latter equation shows that the joint distribution of mean-field variables approaches a Gaussian one, with asymptotically (as $N\to\infty$) independent components $\mathbf{S}_n$. Each $\mathbf{S}_n$ obeys a central limit theorem; its expectation vanishes, and its covariance matrix is scalar, with trace

$$\langle S_n^2\rangle=2\left\langle\left(\frac{1}{N}\sum_{j=1}^{N}q_j\cos k_nx_j\right)^2\right\rangle=\frac{2Q^2N^{-1}}{2+\beta Q^2V_n}+O(N^{-2}). \tag{28}$$

This distribution of $\mathbf{S}_n$ implies that the local field (10) at any point x has a Gaussian distribution, with $\langle A(x)\rangle=0$ and

$$\langle A(x)^2\rangle=N^{-1}Q^2\sum_n k_n^2V_n^2(2+\beta Q^2V_n)^{-1}+O(N^{-2}). \tag{29}$$

The structure factors, i.e., the Fourier coefficients of the correlation function (23), are obtained in the same way.

For a single-species system $(Q=q)$, the fields' responses (24) to a test particle (q) at the origin

$$S_n^{\text{eff}}=\frac{q}{N}\frac{2}{2+\beta Q^2V_n} \tag{30}$$

shows that the screening is more efficient for the larger coupling constants V_n and for the smaller temperatures β^{-1}. One checks that

$$C(y)=L^{-1}-\beta L^{-1}(N-1)^{-1}Q^2V(y)+O(\beta^2)$$
$$=L^{-1}\exp\left(-\frac{\beta Q^2V(y)}{N-1}\right)+O(\beta^2) \qquad (31)$$

in the high-temperature limit ($\beta\to 0$). The repulsion between particles ensures that the Fourier components C_n and $\mathbf{S}_n$ depend only on βV_n, independently of each other. In particular, the fluctuations of $\mathbf{S}_n$ are exactly those of the ideal gas for all n such that $V_n=0$; fluctuations are smaller than for the ideal gas for n such that $V_n>0$. The reduction factor depends on β in such a way that one recovers the ideal gas in the high-temperature limit whereas $N\langle S_n^2\rangle=O(T)\to 0$ in the low-temperature limit, in agreement with the ground-state study.

The diversity of species (various q_j or m_j) does not much affect the thermodynamic behavior: the partition function and thermodynamic potentials depend only on the (quadratic) mean charge Q and the (geometric) mean mass $M=\exp(N^{-1}\Sigma_j \ln m_j)$. The free energy of our model and its entropy per particle are

$$F=F_0+\frac{T}{N}\sum_n \ln\left(1+\frac{Q^2V_n}{2T}\right), \qquad (32)$$

$$S=S_0-\frac{F-F_0}{T}+\frac{T}{N}\sum_n \frac{V_n}{V_n+2T} \qquad (33)$$

where $F_0=-(T/2)\ln(2\pi MT)-LT$ is the ideal gas free energy, and S_0 is the ideal gas entropy. One computes that the contribution of the repulsive interaction increases the free energy, the internal energy $U=F+TS$, and the heat capacity $T\partial S/\partial T$, and reduces the entropy. These formulas hold for all models with $V_n\geqslant 0\,\forall n$; these models have positive energy and undergo no phase transition.

For coefficients $V_n=K'n^{-p}$ for all $n>0$, and a single species ($m_j=M$, $q_j=Q=1$), the partition function is found explicitly:

$$L^{-N}Z_V=\prod_{n=1}^{\infty}\left(1+\frac{\beta K'}{2n^p}\right)^{-1}=\prod_{q=1}^{p}\Gamma(1+z^q), \qquad (34)$$

with $z=(\beta K'/2)^{1/p}e^{2\pi i/p}$ and Euler's gamma function Γ. A special case is the jellium [24]

$$L^{-N}Z_V=\frac{\pi\beta K'}{2\sinh(\pi\beta K'/2)}. \qquad (35)$$

The mean fields have independent isotropic Gaussian distributions with

$$N\langle S_n^2\rangle=\frac{2}{N}\left\langle\left(\sum_{j=1}^{N}\cos k_n x_j\right)^2\right\rangle=(1+\lambda_D^{-2}k_n^{-2})^{-1} \qquad (36)$$

according to Eq. (28). The Debye length $\lambda_D=Q^{-1}(L\epsilon_0/\beta)^{1/2}$ takes into account the normalization of the Coulomb interaction by N^{-1} in the mean-field limit. In the limit ($L\to\infty$, with N/L and λ_D fixed), the spectrum (36) becomes flat, as it must for the plasma. Furthermore, the screened local field (10) associated to a particle at $x_1=0$,

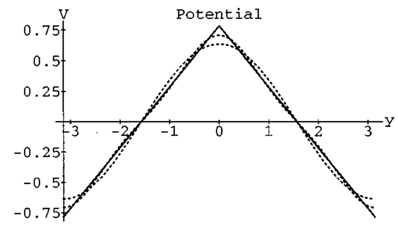

FIG. 1. Pair interaction potential $V(y)$ for Lenard-Prager plasma ($s=\infty$: full line) and for truncations to first Fourier components ($s=1$: dots, $s=3$: dashed line), with $q=\epsilon_0=1$, $L=2\pi$.

$$A(y)=\frac{q}{2N\varepsilon_0}\sum_{r=-\infty}^{\infty}e^{-|y-rL|/\lambda_D}\mathrm{sgn}(y-rL), \qquad (37)$$

and the correlation function

$$C(y)=\frac{N}{L(N-1)}-\frac{1}{2(N-1)\lambda_D}\sum_{r=-\infty}^{\infty}e^{-|y-rL|/\lambda_D}, \qquad (38)$$

exhibit the well-known exponential decay due to Debye screening.

Similarly, the Lenard-Prager plasma [21,22,28,29] is recovered in the limit $s\to\infty$, with coefficients $V_n=K'n^{-2}$ for odd n, and $m_j=M$ and $q_j=Q=1$ (using the conjugation symmetry of odd-n interactions). The partition function is

$$L^{-N}Z_V=\prod_{m=1}^{\infty}\left(1+\frac{\beta K'}{2(2m-1)^2}\right)^{-1}=\mathrm{sech}(\pi\sqrt{\beta K'/8}). \qquad (39)$$

The even Fourier fields $\mathbf{S}_n$ have a temperature-independent distribution with $N\langle S_n^2\rangle=1$, while the odd fields are distributed according to Eq. (36). The correlation function reads

$$C(y)=\frac{1}{L}-\frac{1}{4(N-1)\lambda_D}\sum_{r=-\infty}^{\infty}(-1)^r\exp\left(-k_D\left|y-\frac{rL}{2}\right|\right), \qquad (40)$$

with $k_D^{-2}=\lambda_D^2=L\epsilon_0/(2\beta Q^2)$.

The factor 1/2 (with respect to jellium) accounts for the fact that the Lenard-Prager plasma behaves like a two-species system, containing N particles in a length $L/2$. Indeed, as the sum over n is restricted to odd values of n, the system has a conjugation symmetry: to the particle at position x, with charge q, mass m, and momentum p, one can associate a ghost particle at position $x+L/2$ with the same mass and momentum, but with opposite charge [7–9]. It is then more intuitive to view the system as made of N particles moving on half a circle $0\leqslant x\leqslant L/2$, with modified boundary conditions such that, when a particle exits (enters) on one side, its antiparticle enters (exits) on the opposite side. Figure 1 displays the potential for the Lenard-Prager model ($s=\infty$) and for its truncation to one ($s=1$, $V_1=2/\pi$) and two ($s=3$, $V_1=2/\pi$, $V_2=0$, $V_3=V_1/9$) Fourier components, with $q=\epsilon_0=1$ and $L=2\pi$.

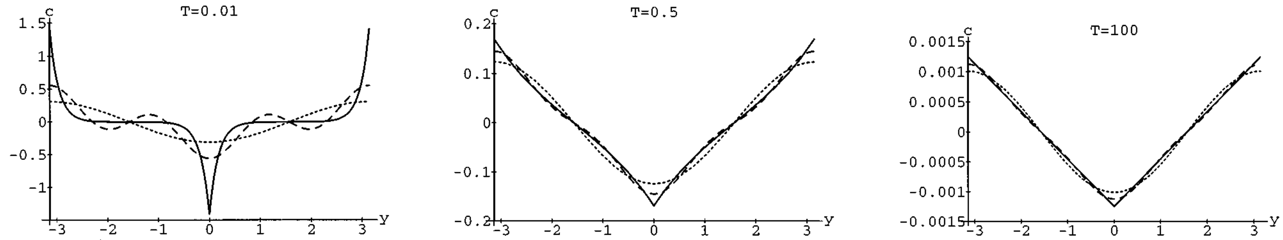

FIG. 2. Nonconstant part $c(y)=(N-1)[C(y)-L^{-1}]$ of the correlation function at given temperatures $T=\beta^{-1}$.

The range of the Debye screening is λ_D and its strength is (almost) inversely proportional to $N_D=2N\lambda_D/L$, which is the expected number of particles in the Debye sphere. The conjugation symmetry of the model in terms of particles shows up in the fact that $C(y)-L^{-1}=L^{-1}-C(y+L/2)$: the Debye repulsion at y for a particle is equivalent to Debye repulsion at $y+L/2$ for its antiparticle.

Figure 2 displays this function for various temperatures for the potentials of Fig. 1. The Debye length is $\lambda_D=\sqrt{\pi T}$. The repelling nature of the interaction appears as correlation functions $C(y)$ are minimum at $y=0$. The Debye length is $\lambda_D=\sqrt{\pi T}$. The repelling nature of the interaction appears as correlation functions $C(y)$ are minimum at $y=0$.

For high temperature ($\beta^{-1}=T=100$), the difference between the model's correlation function $C(y)$ and the ideal gas one [constant L^{-1} in Eq. (31)] is small, and $C(y)=-\beta Q^2V(y)/(N-1)$ to first approximation, in agreement with Eq. (31). The truncations to $s=1$ and 3 of the interaction generate almost the same correlation function as the full sawtooth (Coulomb) potential $s\to\infty$. The shape of C is very close to that of V, and C has almost everywhere the same order of magnitude: there is almost no screening ($\lambda_D \gg L$).

For $T=1/2$, i.e., moderate balance between kinetic and potential energies, the departure of C from L^{-1} becomes larger; screening is still weak, as C still has essentially the shape of V.

For $T=0.01$, the three curves have a significant amplitude, corresponding to a significant effect of the interaction on the particle relative positions. Screening effect is clear as the correlation functions are much closer to 0 away from $y=0$; screening has damped the long-wavelength ($k_n \leqslant k_D$) components of the Coulomb interaction. The $s\to\infty$ correlation function C exhibits a marked peak at $y=0$, and differs significantly from the $s=1$ and 3 correlation functions: the short-range contributions are not screened, and are then of the same order of magnitude as the long-range contributions. This is clear from $C_n=(k_n^2+k_D^2)^{-1}$, which follows from Eqs. (36) and (23).

IV. FERROMAGNETIC MODELS

When some coupling constants V_n are negative, the previous section estimates of Z_V and Ψ no longer hold. As Eq. (19) shows that the dominant contribution to Z_V in the limit $N\to\infty$ results from the neighborhood of the maximum of $\mathcal{J}$, let us estimate how this maximum changes with β and the V_n's.

Let $\gamma_m=-c_m$ for those negative coupling constants and use subscript m instead of n for them. Then, for vectors $\mathbf{u}\in\mathbb{R}^{2s}$, one finds, with $\mathbf{u}_n=(u_n\cos\theta_n, u_n\sin\theta_n)$,

$$\left|\mathcal{J}\left(q\frac{\boldsymbol{\sigma}}{N}+2q\sqrt{c/N}\mathbf{u}\right)\prod_{n=1}^{s} e^{-u_n^2/N}\right|$$

$$\leqslant \frac{1}{2\pi}\int_{S_{2\pi}}\prod_m e^{-u_m^2/N+2q\sqrt{\gamma_m/N}u_m\cos(mx'-\theta_m)}dx' \tag{41}$$

$$\leqslant \frac{1}{2\pi}\int_{S_{2\pi}}\prod_m e^{-u_m^2/N+2q\sqrt{\gamma_m/N}u_m\cos(mx')}dx' \tag{42}$$

$$\leqslant \prod_m e^{-u_m^2/N} I_0(2q\kappa_m\sqrt{\gamma_m/N}u_m)^{1/\kappa_m}, \tag{43}$$

where the latter expression follows from Hölder's inequality for any choice of κ_m such that $\kappa_m\geqslant 1(\forall m)$ and $\Sigma_m\kappa_m^{-1}=1$; I_0 is the modified Bessel function. The equality holds in Eqs. (41) and (42) for $\boldsymbol{\sigma}=\mathbf{0}$, with $\mathbf{u}_n=\mathbf{0}$ if $V_n\geqslant 0$ and $\theta_m=0\forall m$.

It is easily seen (because $I_0(z)\leqslant e^{z^2/4}$) that the upper bound (43) has a unique maximum, at $\mathbf{u}=\mathbf{0}$, provided that $\kappa_m q^2|\gamma_m|<1$ for all m. As equality holds in (41)–(43) for $\mathbf{u}=\mathbf{0}$ and $\boldsymbol{\sigma}=\mathbf{0}$, this implies that Z_V and Ψ are given by Eqs. (25), (26), and (27) provided that $\beta<\beta_c$; in the single-species case $q=1$, this bound yields $\beta_c=-2/(\Sigma_m V_m)$.

The high-temperature regime is thus analogous to the plasma case, except that the expected mean-field intensities are enhanced for Fourier components with $V_m<0$; antiscreening may occur in the correlation function. Our argument only yields a lower estimate $T_c=\beta_c^{-1}$ for the transition temperature: numerical simulations show that this regime may extend to the critical temperature $\beta_c^{-1}=T_c=\sup_m(-Q^2V_m)/2$, which means that the maximum of Eq. (41) is also at the origin for $-2/(\Sigma_m Q^2V_m)\leqslant\beta<\beta_c$. Physically, if one treats the Fourier components $\mathbf{S}_m$ as distinct fields [each of which could condensate at its own temperature $-2/(Q^2V_m)$], then one could say that these components with different wave numbers do not cooperate to raise the temperature of the phase transition.

Note that the phase transition occurs even if the two-particle interaction potential $V(y)$ has a single maximum at $y=0$ and a single minimum at $y=L/2$, which is repulsive on the circle: the transition is indeed a collective effect associated with the mean-field model.

Inequalities (41)–(43) predict the phase transition exactly if there is a single negative coefficient V_m. Indeed, for $\boldsymbol{\sigma}=\mathbf{0}$ and for $\mathbf{u}$ having only one nonzero component ($\mathbf{u}_m$),

$$\mathcal{J}(2q\sqrt{c/N}\mathbf{u})=I_0(2q\sqrt{\gamma_m/N}u_m). \tag{44}$$

In this case, which includes the simple ferromagnetic model [3,13,14,16] with $V_1<0$ and $V_n=0$ for $n>1$, the rotator system has partition function (26) and characteristic function (27) for all temperatures above $T_c=-Q^2V_m/2$, for which Eq. (26) actually diverges; the (anti-)screening (30) *enhances* the bare field. At T_c, this system undergoes a second-order phase transition with order parameter S_m.

Below the critical temperature, the mean-field Fourier components are no longer independent: components $\mathbf{S}_n$ with n not divisible by m are $O(N^{-1/2})$, but $S_n=O(S_m^r)$ and $\phi_n\approx r\phi_m$ for $n=rm$; accordingly, the correlation function is modulated with period L/m. A similar result, with emphasis on the analogy with Jeans instability of self-gravitating systems, was obtained by Inagaki [13] through maximizing entropy.

V. CONCLUSIONS

We computed the partition function, thermodynamic potentials, field distribution and correlation function of the generic family of one-dimensional systems of N particles interacting pairwise via their mean fields. Our method is quite classical, and involves neither the integral operators used in the exact solutions available in the literature [23–25,21,28–30] nor recurrence relations [31] for the grand partition function.

If all mean fields contribute positively to the interaction energy, i.e., in the fully antiferromagnetic case, the fields $\mathbf{S}_n$ obey central limit theorems and behave independently in the limit $N\rightarrow\infty$ at any temperature: no phase transition occurs in this family. We recover the observation that the force $qA(x)$ experienced by a test particle at position x is a Gaussian stochastic process with respect to x, and a white noise [29] in the case of a plasma in the limit $L\rightarrow\infty$. For the plasma, the correlation function and the (screened) effective force field generated by a particle are obtained analytically: they decay exponentially, on the well-known characteristic Debye length.

If a single mean field contributes negatively to the interaction energy, the system undergoes a second-order phase transition with this field as order parameter; the high-temperature regime has the same thermodynamics and field distributions as the fully antiferromagnetic case. Debye screening is replaced by enhancement for the corresponding field component, with the mechanism of the Jeans instability. The case with several mean fields contributing negatively has the same high-temperature regime; the low-temperature regime deserves a separate study: for instance, one cannot exclude the occurrence of several phase transitions.

The present work makes no prediction on the time evolution of our systems: equilibrium statistical mechanics cannot compute relaxation times or describe the evolution of fields, or of turbulent structures, as observed, e.g., in the XY model (or $s=1$ rotator model) [3–6,8,16,32]. However, it provides a background of predictions enabling an interesting comparison with the dynamical behavior of the models under consideration. These questions will be discussed in forthcoming papers.

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